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Path-following energy optimization in unilateral contact problems

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Abstract. Path-following (load incrementation) methods are studied in this paper for elastostatic analysis problems with unilateral contact relations in the framework of a large displacement theory by means of the parametric optimization techniques. Finite element discretization yields sparse polynomial optimization problems with equality and inequality constraints. For such sparse problems generically appearing singularities along the path of solutions are completely classified. Perturbations involving only a minimal number of parameters are shown to be sufficient to guarantee these generic situations. This clarifies stability and uniqueness questions for the solution along the examined path.

Key words: pathfollowing, parametric optimization, unilateral contact problems, nonsmooth mechanics

1 Introduction

Inequality and equality constrained structural analysis problems frequently arise in civil and mechanical engineering applications. The unilateral contact problem constitutes a simple example in elasticity with inequality constraints on the primary variables (displacements) of a structure. The non-penetration assumption between an elastic body and a rigid support, or between two elastic bodies, introduces inequalities in the problem (see among others [8], [19]).

The numerical treatment of inequality mechanics' problems is usually done by means of finite element discretization of the structure and iterative solution techniques based on load incrementation. The mechanical analogon of this procedure is easily conceived: the external action (load or imposed displacements) is applied gradually on the elastic structure in a quasistatic way and the generated equilibrium path is traced numerically, until the end of the prescribed loading sequence or the yielding of the structure if no solution of the posed problem exists. This procedure has been thoroughly investigated for smooth unconstrained problems (cf. the recent expositions in [22], [7], [4], [6]) and can be extended to tackle inequality constrained problems as well in an active index set methodology. Theoretical questions pertaining to this extension are investigated in this paper.

Since large displacement elastostatic unilateral contact problems can be formulated (under certain assumptions concerning the finite element approximation and the linearizations used, see e.g. [3]) as a nonconvex, inequality constrained optimization problem for the potential energy function, a critical study of the procedures used in nonlinear mechanics in the framework of parametric optimization theory is considered to be relevant.

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In this paper the path-following procedure for the discretized problem will be identified with a one-parameter optimization problem. Subsequently existence, stability and possible bifurcation of the solution along this path will be studied by applying results of generic, one-parametric transformations of optimization problems [9], [10], [11], [21]. In particular the question of realizability of this solution path will be answered, i.e. conditions will be formulated so that the procedure can trace the path of (even unstable) solutions of the problem. In this sense this work complements and generalizes the study of [14]. Extension of the path-following procedure through segments of the critical point set which violate the nonpenetration assumption and thus do not correspond to physically realizable configurations of the mechanical structure will be proposed here in the hope to avoid discontinuities in the equilibrium points'path. The latter possibility has not been investigated in computational mechanics for inequality constrained problems, although its counterpart for smooth problems has been widely accepted [7].

It should be noted here that sparsity of the parametric optimization problem, as it occurs in inequality mechanics applications which are discretized by the finite element method, is taken into account in this paper. The results of section four can thus be considered to be specializations of the ones given in [9] for the fully occupied C^3 problem, to sparse problems. To the best of the authors knowledge, this is the first attempt where genericity theory for parametric optimization meets sparse problems, as they arise in real–life applications.

The layout of the paper is as follows: the formulation of a discretized nonlinear energy optimization problem for a unilateral structure will be given in section two. Questions posed along a path-following technique and mechanical interpretation of the effects that are expected to occur and their impact on the solution strategy will be addressed in section three. Stability, uniqueness and existence of the solution along the one-parametric path considered will be discussed in section four, based on the theoretical study of [9]. The proof of the genericity theorem is given in the last section.

2 Nonlinear energy optimization for discretized unilateral contact problems

2.1 The basic relations of the problem

Let the discrete stress and strain variables of a discretized structure be assembled in the l-dimensional vectors $\mathbf{e}$ and $\mathbf{s}$ respectively. Let $\mathbf{u}$ be the n-dimensional vector of nodal displacements and $\mathbf{p}$ the energy corresponding nodal loading vector. The loading is parametrized by a scalar parameter t. The whole structure is referred to a Cartesian, right-handed coordinate system. All quantities will be referred to the initial (undeformed) configuration of the elastic body (total Lagrangian formulation in the terminology of nonlinear mechanics). Let discrete contact tractions and corresponding displacements be assembled in the s-dimensional vectors $\mathbf{S}_\mathbf{n}^\mathbf{s}$ and $\mathbf{u}_\mathbf{n}^\mathbf{s}$ respectively – for that part of the boundary of the structure where unilateral contact effects are expected to occur. On the part of the boundary where displacement boundary conditions (classical bilateral support) will be assigned the discretized traction and displacement variables are assembled in the q-dimensional vectors $\mathbf{S}_\mathbf{n}^\mathbf{u}$ and $\mathbf{u}_\mathbf{n}^\mathbf{u}$ respectively.

The following relations govern the large displacement, discretized elastostatic analysis problem (see [19], [14], [20] among others):

Compatibility relations

$$\bar{\mathbf{e}} = \begin{bmatrix} \mathbf{e} \\ \mathbf{u}_\mathbf{n}^\mathbf{s} \\ \mathbf{u}_\mathbf{n}^\mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{G}(\mathbf{u}) \\ \mathbf{g}(\mathbf{u}) \\ \mathbf{h}(\mathbf{u}) \end{bmatrix} = \overline{\mathbf{G}}(\mathbf{u}) \tag{1}$$

where $\overline{\mathbf{G}}(\mathbf{u})$ is the nonlinear displacement- deformation compatibility operator ($\mathbf{G} : I\!R^n \rightarrow I\!R^l$, $\mathbf{g} : I\!R^n \rightarrow I\!R^s$, $\mathbf{h} : I\!R^n \rightarrow I\!R^q$).

Equilibrium equations

$$D_u\overline{\mathbf{G}}^\mathbf{T}(\mathbf{u})\bar{\mathbf{s}} = \begin{bmatrix} D_u\mathbf{G}(\mathbf{u})^\mathbf{T} & D_u\mathbf{g}(\mathbf{u})^\mathbf{T} & D_u\mathbf{h}(\mathbf{u})^\mathbf{T} \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ -\mathbf{S}_\mathbf{n}^\mathbf{s} \\ -\mathbf{S}_\mathbf{n}^\mathbf{u} \end{bmatrix} = \mathbf{tp} \tag{2}$$

where D_u denotes the first derivative of the nonlinear operator and superscript T denotes the transposed matrix or vector.

Hyperelastic constitutive law

$$\mathbf{s} = \mathbf{D_e}\Phi(\mathbf{e}) \tag{3}$$

where $\Phi(\mathbf{e})$ is an appropriately defined convex and smooth potential function which in the case of a linear elastic material is a quadratic function, i.e. the law takes in this case the form (Hookean material) $\mathbf{s} = \mathbf{K}_0\mathbf{e}$ or $\Phi(\mathbf{e}) = \frac{1}{2}\mathbf{e}^T\mathbf{K}_0\mathbf{e}$ with $\mathbf{K_0}$ the natural stiffness matrix (assumed symmetric and positive semi-definite).

unilateral contact boundary conditions

$$\begin{aligned} -\mathbf{S}_\mathbf{n}^\mathbf{s} \in \partial\mathbf{I}_{\mathbf{U}_\mathbf{ad}(\mathbf{u})}(\mathbf{u}) &= \mathcal{N}_{\mathbf{U}_\mathbf{ad}(\mathbf{u})}(\mathbf{u}) = \\ = \{ -\mathbf{S}_\mathbf{n}^\mathbf{s} &= \mathbf{v}^\mathbf{T}\mathbf{D_u}\mathbf{g}(\mathbf{u}),\ \mathbf{v} \in \mathbb{R}^\mathbf{s}, \mathbf{v_i} \geq \mathbf{0},\ i = 1,\ldots,s \} \end{aligned} \tag{4}$$

where ∂ denotes the subdifferential operator of convex analysis, $\mathbf{I_K}$ is the indicator function of the set K, $\mathcal{N}_K$ denotes the normal cone to the set K and $U_{ad}(\mathbf{u})$ is the set of admissible displacements of the structure as restricted by the nonlinear, no-penetration kinematic restrictions imposed by the unilateral contact effects:

$$U_{ad}(\mathbf{u}) = \{ \mathbf{u} \in \mathbb{R}^\mathbf{n} : \ \mathbf{g}(\mathbf{u}) \leq \mathbf{0} \} \tag{5}$$

Bilateral (support) boundary conditions

$$\mathbf{h}(\mathbf{u}) = \mathbf{0} \tag{6}$$

2.2 ENERGY MINIMIZATION PROBLEM

A potential energy minimization problem in a total Lagrangian setting of the mechanical problem (i.e. everything is referred to the initial, undeformed configuration), and an incremental variational formulation, more suitable for an updated Lagrangian setting, will be outlined in this section – together with the approximations used in computational mechanics for the derivation of various order mechanical theories. The purpose of this paragraph is to underline the connection between energy optimization and nonlinear structural analysis, without any claim for completeness.

Energy formulation

The total potential energy minimization problem is of the form: Find displacements of the structure $\mathbf{u} \in \mathbb{R}^{\mathbf{n}}$ such as to solve the *constrained minimization problem*:

$$\Pi(\mathbf{u}) = \min_{\mathbf{v} \in \mathbf{U_{ad}}} \{\mathbf{\Pi}(\mathbf{v}) = \mathbf{\Pi_{int}}(\mathbf{v}) - \mathbf{\Pi_{ext}}(\mathbf{v})\}$$

with

$$\mathbf{v} \in U_{ad} = \{ \mathbf{v} \in \mathbb{R}^n \mid h_i(\mathbf{v}) = 0,\ g_j(\mathbf{v}) \leq 0 \text{ for all } 1 \leq i \leq q,\ 1 \leq j \leq s\}. \quad (7)$$

Here $\Pi_{int}(\mathbf{v}) = \mathbf{\Pi_{int}}(\mathbf{e}(\mathbf{v}))$ is the internally stored elastic energy of the structure, which for a linearly elastic material takes the form:

$$\Pi_{int}(\mathbf{v}) = \{\frac{\mathbf{1}}{\mathbf{2}}\mathbf{e}^{\mathbf{T}}\mathbf{K_0}\mathbf{e} \mid \mathbf{e} = \mathbf{G}(\mathbf{u})\}. \quad (8)$$

The function $\Pi_{ext}(\mathbf{v})$ is the potential energy of the externally applied loading, which is assumed to be conservative, i.e. according to (2):

$$\Pi_{ext}(\mathbf{v}) = -t\ \mathbf{p}^T\mathbf{u} \quad (9)$$

Approximation of the displacement-deformation compatibility equation (first relation in (1)) by using Taylor series expansion techniques, restriction to certain lower order terms in this expansion and substitution in (8), (7) results in various order approximation theories for the mechanical problem. For instance in a first-order theory (i.e. small displacement assumption) $\mathbf{G}$ is a linear function of $\mathbf{u}$, i.e. $\mathbf{G} = \mathbf{G_1}\mathbf{u}$. If first order terms are used in the aforementioned expansion and in the subsidiary equality and inequality constraints of the problem, we get the *quadratic, linearly constrained minimization problem*: (written here under the additional assumption that $G(\mathbf{u_0}) = 0$, for notational simplicity)

$$\Pi(\mathbf{u}) = \min_{\mathbf{v} \in \mathbf{U_{ad}^{lin}}} \{\mathbf{\Pi}(\mathbf{v}) = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{v}^{\mathbf{T}}[\mathbf{G_1^T}\mathbf{K_0}\mathbf{G_1}]\mathbf{v} - \mathbf{t}\mathbf{p}^{\mathbf{T}}\mathbf{v}\} \quad (10)$$

with

$$U_{ad}^{lin} = \{\mathbf{v} \in \mathbb{R}^{\mathbf{n}} \mid h_i(\mathbf{u_0}) + \mathbf{D_u h_i}(\mathbf{u_0})\mathbf{v} = \mathbf{0},\ \mathbf{1} \leq \mathbf{i} \leq \mathbf{q}, \\ g_j(\mathbf{u_0}) + \mathbf{D_u g_j}(\mathbf{u_0})\mathbf{v} \leq \mathbf{0},\ \mathbf{1} \leq \mathbf{j} \leq \mathbf{s}\} \tag{11}$$

where $\mathbf{K} = \mathbf{G_I^T K_0 G_1}$ is the assembled stiffness matrix of the structure. The optimality conditions for problem (10) give rise to the classical *linear variational inequality problem* (cf. e.g. [8], [18], [19]): Find $\mathbf{u} \in \mathbf{U_{ad}^{lin}}$ such that:

$$\mathbf{u^T K}(\mathbf{v} - \mathbf{u}) - \mathbf{t p^T}(\mathbf{v} - \mathbf{u}) \geq \mathbf{0},\ \forall \mathbf{v} \in \mathbf{U_{ad}^{lin}} \tag{12}$$

Remark: Note that the above mentioned linearization must be consistent with the fact that our problem admits a potential, since otherwise nonsymmetric matrices arise in the expression of $\Pi(\mathbf{u})$ (cf. e.g. [3], [20]). Moreover different order approximations for e.g. displacement and rotation variables in a nonlinear elastic structure lead to the formulation of various models in mechanics (cf. various shell and plate theories). These questions will not be discussed here. □

Incremental formulation

Incremental energy optimization problems can be formulated by using standard linearization techniques and taking into account the inequality and complementary relations (cf. [19], [14], [2], [22]). In the sense ot the previously mentioned approximation scheme, using second order terms in kinematic linearization, a linear material law and first order terms in the expansion of equality and inequality constraints, the following incremental problem can be formulated (for notational simplicity equality constraints are not considered):

Let the stress, displacement and deformation state of the structure be given for loading level t (i.e. $\bar{\mathrm{s}}, \bar{\mathrm{e}}, \bar{\mathrm{u}}, -\overline{\mathbf{S}}_{\mathbf{n}}^{\mathbf{s}}, \mathcal{J}_0(\bar{\mathrm{u}})$), where $\mathcal{J}_0(\mathbf{u})$ denotes the set of active indices in $\mathbf{u}$, i.e.

$$\mathcal{J}_0(\mathbf{u}) := \{j \in \{1, \ldots, s\} \mid g_j(\mathbf{u}) = 0\}. \tag{13}$$

Let us formulate the problem that describes the incremental changes of the above quantities (denoted by $\dot{\mathrm{s}}, \dot{\mathrm{e}}, \dot{\mathrm{u}}, -\dot{\mathbf{S}}_{\mathbf{n}}^{s}$) as the load level is increased by δt. To this end the defining relations of the problem are differentiated with respect to t, assuming that the gradients of the involved functions exist (where $\overline{u}$ is the solution at the beginning of the time interval δt, i.e. at $\bar{t}$).

$$\dot{\mathbf{p}}dt = D_{\mathbf{u}}\mathbf{G}(\bar{\mathrm{u}})^T \mathbf{K}_0 \frac{1}{2} D_{\mathbf{uu}}\mathbf{G}(\bar{\mathrm{u}})\dot{\mathrm{u}}dt\ \dot{\mathrm{u}}dt + D_{\mathbf{u}}\mathbf{G}(\bar{\mathrm{u}})^{\mathbf{T}}\mathbf{K_0 D_u G}(\bar{\mathrm{u}})\dot{\mathrm{u}}dt \tag{14}$$

$$+ \mathrm{s}^T D_{\mathbf{uu}}^2 \mathbf{G}(\bar{\mathrm{u}})\dot{\mathrm{u}}dt + \sum_{j \in \mathcal{J}_0(\mathbf{u})} -\overline{\mathbf{S}}_{\mathbf{n_j}}^{\mathbf{s}} D_{\mathbf{uu}}\mathbf{g_j}(\bar{\mathrm{u}})\dot{\mathrm{u}}dt - D_{\mathbf{u}}\mathbf{g}(\bar{\mathrm{u}})^{\mathbf{T}}\dot{\mathbf{S}}_{\mathbf{n}}^{\mathbf{s}}dt + \ldots$$

For the unilateral contact constraints we consider first the following complete subdivision of the index set of inequality constraints:

The *strongly active* index set, which is denoted by

$$\mathcal{I}_1 \; = \; \{j \in \{1, \ldots s\} \text{ such that } g_j(\overline{\mathbf{u}}) = 0 \text{ and } -\mathbf{S}^{\mathrm{s}}_{\mathbf{n_j}}(\overline{\mathbf{u}}) > \mathbf{0}\} \tag{15}$$

the *semi-active* index set

$$\mathcal{I}_2 \; = \; \{j \in \{1, \ldots s\} \text{ such that } g_j(\overline{\mathbf{u}}) = 0 \text{ and } -\mathbf{S}^{\mathrm{s}}_{\mathbf{n_j}}(\overline{\mathbf{u}}) = \mathbf{0}\} \tag{16}$$

and the *inactive* index set

$$\mathcal{I}_3 \; = \; \{j \in \{1, \ldots s\} \text{ such that } g_j(\overline{\mathbf{u}}) < 0 \text{ and } -\mathbf{S}^{\mathrm{s}}_{\mathbf{n_j}}(\overline{\mathbf{u}}) = \mathbf{0}\} \tag{17}$$

with cardinalities equal to s_1, s_2, s_3 respectively (with $s_1 + s_2 + s_3 = s$).

The incremental quantities (e.g. $\dot{\mathbf{u}}^s_n dt$) can be considered as directional derivatives, along the direction defined by the loading vector, of the solution of an inequality constrained energy optimization problem (compare with the previous paragraph; see e.g. [14] and [5] for details on the mechanical and mathematical issues respectively). So the rate problem is formulated as an *incremental energy optimization problem*:

$$\min_{\dot{\mathbf{u}}} \{\frac{1}{3}\dot{\mathbf{u}}^T[K_{ijk}(\overline{\mathbf{u}})]\dot{\mathbf{u}}\dot{\mathbf{u}} \; + \; \frac{1}{2}\dot{\mathbf{u}}^T[\mathbf{K_T} + \mathbf{K_s} + \mathbf{K^c_s}]\dot{\mathbf{u}} \; - \; \mathring{\mathbf{p}}^{\mathbf{T}}\dot{\mathbf{u}} \; \}$$

such that

$$\dot{\mathbf{u}}^s_{n_i} dt = D_u g_i(\overline{\mathbf{u}})\dot{\mathbf{u}}dt \leq 0, \; \forall i \in \mathcal{I}_2, \; \dot{\mathbf{u}}^s_{n_i} dt = D_u g_i(\overline{\mathbf{u}})\dot{\mathbf{u}}dt = 0, \; \forall i \in \mathcal{I}_1 \tag{18}$$

In (18) the third order stiffness matrix $K_{ijk}(\overline{\mathbf{u}})$ is used, where

$$K_{ijk}(\overline{\mathbf{u}}) = D_u \mathbf{G}(\overline{\mathbf{u}})^{\mathbf{T}}\mathbf{K_0}\frac{\mathbf{1}}{\mathbf{2}}\mathbf{D_{uu}G}(\overline{\mathbf{u}}) \tag{19}$$

the initial strain and stress stiffness matrices $\mathbf{K_T}$ and $\mathbf{K_s}$ are defined (cf. [22], [2])

$$\mathbf{K_T} = \mathbf{D_u G}(\overline{\mathbf{u}})^{\mathbf{T}}\mathbf{K_0 D_u G}(\overline{\mathbf{u}}), \; \mathbf{K_s} = \mathbf{s^T D_{uu} G}(\overline{\mathbf{u}}) \tag{20}$$

and an additional contribution $\mathbf{K^c_s}$ to the initial stress stiffness matrix appears that depends on the initial contact stresses and the curvature of the boundary surface, namely (cf. [14]):

$$\mathbf{K^c_s} = \sum_{\mathbf{j} \in \mathcal{I}_2} -\overline{\mathbf{S}}_{\mathbf{n_j}}\mathbf{D_{uu}g_j}(\overline{\mathbf{u}}) \tag{21}$$

In (21) an appropriate renumbering of the displacement variables has been assumed such that the first s_1 variables of $\overline{\mathbf{u}}$ are equal to the tangential displacements arising at the active set of the unilateral contact constraints.

We shall not discuss more modelling aspects here – we shall use instead a general model formulation of the energy optimization problem, which can then be fitted to every computational mechanics model considered.

Model formulation

In the sequel the following general discrete potential energy optimization problem will be considered as a model problem (analogously to [16], [17] for the unconstrained case): Find the (even local) minimization point(s) $\mathbf{u} \in \mathbf{U_{ad}}$ (i.e. stable equilibrium solutions of the structure) or, generally, the critical points (i.e. stable and unstable equilibrium points of the structure) of the potential energy minimization problem

$$\Pi(\mathbf{u},t) = \min_{\mathbf{v} \in U_{ad}} \Pi(\mathbf{v},t) \tag{22}$$

with

$$\begin{aligned} \Pi(\mathbf{u},t) &= \mathbf{W}(\mathbf{u}) - \mathbf{V}(\mathbf{u},t) = \mathbf{A}^T\mathbf{u} + \mathbf{u^T B u} + \mathbf{u^T C u u} - t\mathbf{p^T u} \\ &= \sum_{i=1}^{n} a_i u_i + \frac{1}{2}\sum_{1\le i,j\le n} b_{ij}u_i u_j + \frac{1}{6}\sum_{1\le i,j,k\le n} c_{ijk}u_i u_j u_k - t\sum_{i=1}^{n} u_i p_i \end{aligned} \tag{23}$$

(with $a_i, b_{ij}, c_{ijk}, p_i \in R$, $b_{i,j} = b_{j,i}$ and $c_{ijk} = c_{jik} = c_{ikj}$ for all $i,j,k \in \{1,\ldots,n\}$) and

$$U_{ad} = \{\mathbf{u} \in I\!R^n \mid h_i(\mathbf{u}) = 0,\ g_j(\mathbf{u}) \geq 0 \text{ for all } 1 \leq i \leq q,\ \leq j = 1 \leq s\}. \tag{24}$$

In (23) index notation has been used for the expression of the potential energy function with respect to a right handed, orthogonal coordinate system. The symmetric second and third order tensors $\mathbf{B}$ (resp. $\mathbf{C}$) have been used with the meaning of appropriate (resp. higher order) stiffness matrices. The vector $\mathbf{A}$ stands for appropriate initial stresses. The nonlinear relations in (24) are assumed to have derivatives of analogous order. This is required by the specific application (e.g. mechanical theory) on hand.

3 One–parametric transition and load–incrementation

The parameter $t \in I\!R^1$ in problem (22 - 24) controls the applied load on the unilaterally constrained elastic structure. Starting from zero level of loading (i.e. $t = 0$), where initial displacements are known $\mathbf{u} = \mathbf{0}$, the equilibrium configuration(s) of the structure can be followed by walking along the path $t \in [0, t_1]$ until the final loading level $t_1\mathbf{p}$ is reached.

If we can determine local minimum points for the potential energy minimization problem along the whole path, then this path leads to a quasistatic variation of the displacement, stress and strain variables of the structure. This implies that snap-through and snap-back effects that could cause departure from the quasistatic behaviour assumption do not arise.

Sometimes certain parts of the path can not be followed in the space of minimizing points, but a continuation in the enlarged space of critical points is possible (i.e. both mechanically stable and unstable equilibrium points of the structure are considered. This latter path can be followed numerically, although certain dynamic transitions are expected to occur in the real structural system if this loading sequence is applied to it [7].

If at some load level $t_2\mathbf{p}$ the one-parametric path of minimum points can not be followed any more, then either our structure collapses at that point (i.e. the problem has no solution or intuitively speaking there does not exist enought supports that could stabilize the analysed structure) or a solution of the problem exists but it can not be reached by path following along the chosen path (parametrization).

One more comment concerning the analysis of the parametric optimization problem that follows is in due course here. Since finite element approximations are used for the discretization and the numerical treatment of mechanical problems, the arising problems are sparse. Sparsity is taken into account in the analysis of the parametric optimization problem that follows. Thus the results of the next sections can be shown as a specialization and a concretization of the results given in [9] for fully occupied C^3–problems to the sparse problems of finite element analysis in mechanics.

4 Generically appearing cases along the parametric path

4.1 Sparse unconstrained case

Let us consider for the moment the unconstrained energy optimization problem (23) with $U_{ad} = I\!R^n$. We assume that no unilateral constraints are prescribed and that sufficient equality constraints have been incorporated in the finite element model such as to prevent every rigid body motion of the structure.

In this case, for each load level $\bar{t}$ the equilibrium configuration $\overline{\mathbf{u}}$ of the structure is a critical point of the energy function (cf. (23))

$$\Pi(\mathbf{u},t) = W(\mathbf{u}) - V(\mathbf{u},t) = \mathbf{A}^T\mathbf{u} + \frac{1}{2}\mathbf{u}^T\mathbf{B}\mathbf{u} + \frac{1}{6}\mathbf{u}^T\mathbf{C}\mathbf{u}\mathbf{u} - t\mathbf{p}^T\mathbf{u}. \quad (25)$$

The point $\overline{\mathbf{u}}$ is called a critical point of $\Pi(\mathbf{u},t)$ for the load level t, if

$$(\mathbf{u},t) \in \Sigma_{crit} = \{(\overline{\mathbf{u}},\bar{t}) \in I\!R^{n+1} \mid D_u\Pi(\overline{\mathbf{u}},\bar{t}) = \mathbf{A} + \mathbf{B}\overline{\mathbf{u}} + \frac{1}{2}\overline{\mathbf{u}}^T\mathbf{C}\overline{\mathbf{u}} - \bar{t}\mathbf{p} = \mathbf{0}\}. \quad (26)$$

The subset of Σ_{crit} where the second derivative ("stiffness matrix") of the potential energy function becomes singular is

$$\Sigma_{deg} = \{(\overline{\mathbf{u}},\bar{t}) \in \Sigma_{crit} \mid D_{uu}\Pi(\overline{\mathbf{u}},\bar{t}) = \mathbf{B} + \mathbf{C}\overline{\mathbf{u}} \text{ is singular }\} \quad (27)$$

For $(\overline{\mathbf{u}},t) \in \Sigma_{crit}$ the point $\overline{\mathbf{u}}$ is called a *degenerate* critical point of $\Pi(\mathbf{u},t)$ for the load level t, if $(\overline{\mathbf{u}},t) \in \Sigma_{deg}$ – else it is called *nondegenerate*.

There is an open and dense set of coefficients for $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{p}$ with the following properties (cf. genericity theorem below)

- Σ_{crit} is a manifold and $\Sigma_{deg} \subseteq \Sigma_{crit}$ is a finite set of isolated points
- for all $(\mathbf{u}, t) \in \Sigma_{deg}$ exactly one eigenvalue of $D_{uu}\Pi(\mathbf{u}, t)$ passes zero and the projection map $\phi_t(\mathbf{u}, t) := t$ restricted to Σ_{crit} has a quadratic maximum or minimum in $(\mathbf{u}, t)$

All points of the set $\Sigma = \Sigma_{crit} \backslash \Sigma_{deg}$ belong to the set of nondegenerate critical points of Π that are designated to be points of Type 1 later on in this paper and in [9]. By the implicit function theorem there exist a path of minima $\mathbf{u}(t)$ for the one-parametric optimization problem, as far as we remain in the set $\Sigma = \Sigma_{crit} \backslash \Sigma_{deg}$ and this path can be traced by classical path-following techniques.

At the points in Σ_{deg} the path of minima (e.g. stable solutions of the mechanical problem) with respect to the parametric optimization problem can not be followed any more. If one considers following this path, the appearance of jumps and their algorithmic realization must be considered (see [9]).

Thus for the unconstrained case the results of the detailed analysis given in the next section extend the results of [12], [13] for fully occupied problems and C^3–functions to *sparse* problems.

4.2 INEQUALITY AND EQUALITY CONSTRAINED SPARSE PROBLEM

For a constrained set of admissible displacements with general nonlinear unilateral and bilateral constraints, certain modifications of the previously described scheme must be considered.

Let us first examine closely the constraint set of problem (23) (note the different definitions of the inequalities in order to comply with the form used in [12], [13])

$$U_{ad} = \{\mathbf{u} \in I\!R^n | h_i(\mathbf{u}) = 0,\ 1 \le i \le q,\ g_j(\mathbf{u}) \ge 0,\ 1 \le j \le s\ \} \tag{28}$$

The equilibrium points $\overline{\mathbf{u}}$ at a load level $\overline{t}$ minimize the energy function $\Pi(.,\overline{t})$ in (23) and (25), i.e.

$$\Pi(\overline{\mathbf{u}}, \overline{t}) = \min\{\Pi(\mathbf{u}, \overline{t}) \mid \mathbf{u} \in U_{ad}\ \} \tag{29}$$

We say that a point $(\overline{\mathbf{u}}, \overline{t}) \in U_{ad}(t)$ satisfies the linear independence constraint qualification (shortly $LICQ$) if the vectors

$$Dh_i(\overline{\mathbf{u}}),\ Dg_j(\overline{\mathbf{u}}),\ \text{for } 1 \le i \le q,\ j \in \mathcal{J}_0(\overline{\mathbf{u}}) \tag{30}$$

are linear independent, where $\mathcal{J}_0(\overline{\mathbf{u}})$ denotes the set of active indices in $\overline{\mathbf{u}}$.

If the condition $LICQ$ holds for all $\mathbf{u} \in U_{ad}$, then the critical points, have to satisfy a Lagrange condition:

$$\Sigma_{crit} := \{(\overline{\mathbf{u}}, \overline{t}) \in U_{ad} \times I\!R \mid \exists \overline{\lambda}_1, \ldots, \overline{\lambda}_q, \overline{\mu}_j \in R,\ \forall j \in \mathcal{J}_0(\overline{\mathbf{u}}) \text{ with} \quad D_u\Pi(\overline{\mathbf{u}}, \overline{t}) = \sum_{i=1}^{q} \overline{\lambda}_i Dh_i(\overline{\mathbf{u}}) + \sum_{j \in \mathcal{J}_0(\overline{\mathbf{u}})} \overline{\mu}_j Dg_j(\overline{\mathbf{u}})\ \} \tag{31}$$

If moreover $\overline{\mu}_j \geq 0$ for all $j \in \mathcal{J}_0$ in (31) for some point $(\overline{\mathbf{u}}, \overline{t})$, then $\overline{u}$ is called a *Karush Kuhn Tucker (KKT) point* at the level t. KKT-points are possible candidates for equilibrium points of the mechanical system, since otherwise there exists $j \in \mathcal{J}_0(\overline{\mathbf{u}})$ such that $\mu_j < 0$, i.e. adhesive traction would arise which is not permitted in our model.

Let for $(\overline{\mathbf{u}}, \overline{t}) \in \Sigma_{crit}$ the *LICQ* be satisfied. If additionally for $\overline{\mu}_j$, $\overline{\lambda}_j$ as in Σ_{crit} the two conditions

$$\text{ND1} \qquad \overline{\mu}_j \neq 0 \ \text{ for all } \ j \in \mathcal{J}_0(\overline{\mathbf{u}}) \tag{32}$$

$$\text{ND2} \qquad D^2_{uu} L(\overline{\mathbf{u}}, \overline{t})|_{T(u)} \text{ is nonsingular} \tag{33}$$

hold then $\overline{\mathbf{u}}$ is called a *nondegenerate* critical point of $\Pi(\,.\,, \overline{t})$ with respect to U_{ad}. Here $L(\mathbf{u}, t)$ denotes the Lagrangian

$$L(\mathbf{u}, \mathbf{t}) = \mathbf{\Pi}(\mathbf{u}, \mathbf{t}) - \sum_{\mathbf{i}=1}^{\mathbf{q}} \overline{\lambda}_\mathbf{i} \mathbf{h}_\mathbf{i}(\mathbf{u}) - \sum_{\mathbf{j} \in \mathcal{J}_0(\mathbf{u})} \overline{\mu}_\mathbf{i} \mathbf{g}_\mathbf{j}(\mathbf{u}) \tag{34}$$

and $T(u)$ the tangent space in $\mathbf{u}$ to the set of admissible displacements U_{ad}, i.e.

$$T(\mathbf{u}) = \bigcap_{i \in \{1, \ldots, q\}} \ker(Dh_i(\mathbf{u})) \cap \bigcap_{j \in \mathcal{J}_0(\mathbf{u})} \ker(Dg_j(\mathbf{u})) \tag{35}$$

The number of positive / negative Lagrange multipliers and positive / negative eigenvalues of $L(\mathbf{u}, t)|_{T(\mathbf{u})}$ characterize the problem locally (cf. [9], p. 203, or [12]).

Via some coordinate transformation there exists a simple normal form for the optimization problem in the neighbourhood of nondegenerate critical points (cf. [9], p.34).

4.3 Generically appearing cases in sparse one-parametric transformation

The polynomials Π, g_j, h_i defining the problem in (21) are sparse in the sense that e.g. $g_j(\overline{\mathbf{u}})$ may not depend on all variables u_i or the Hessian $D_{uu}\Pi$ has nonzero entries only close to the diagonal (banded matrix). The following sparse subspaces of polynomials take into account the specific form of $(\Pi, g_1, \ldots, g_s, h_1, \ldots, h_q)$.

We abbreviate

$$\mathbb{R}_d[x_1, \ldots x_n] = \{p : \mathbb{R}^n \to \mathbb{R}^1 | p \text{ polynomial in } x_1, \ldots, x_n, \text{ degree}(p) \leq d\}.$$

Let $\mathcal{P}$ be a linear subspace of $\mathbb{R}_d[u_1, \ldots, u_n, t]$ and $\mathcal{G}_j$, $\mathcal{H}_i$ linear subspaces of $\mathbb{R}_d[u_1, \ldots, u_n]$ for all $1 \leq i \leq q$, $1 \leq j \leq s$ with the following properties:

For all $\Pi(\mathbf{u}, t) \in \mathcal{P}$, $g_j(\mathbf{u}) \in \mathcal{G}_j$, $h_i(\mathbf{u}) \in \mathcal{H}_i$ and $\alpha, \beta, \delta \in R$ with $|\alpha|, |\beta|, |\delta|$ sufficiently small holds

i) $\Pi(\mathbf{u}, t) + \alpha \sum_{i=1}^{n} u_i^2 + \beta u_i \in \mathcal{P}$ and

ii) $g_j(\mathbf{u}) + \delta \in \mathcal{G}_j$ and $h_i(\mathbf{u}) + \delta \in \mathcal{H}_i$ for all $1 \leq i \leq q$, $1 \leq j \leq s$.

All the functions $(\Pi, g_1, \ldots, g_s, h_1, \ldots, h_q)$ we consider below are elements of the finite dimensional space

$$\mathcal{M} = \mathcal{P} \times \mathcal{G}_1 \times \ldots \times \mathcal{G}_s \times \mathcal{H}_1 \times \ldots \times \mathcal{H}_q.$$

Remark The definition of the problem space $\mathcal{M}$ is much less restrictive than the problem formulation in (23). There are no symmetry conditions in the definition of $\mathcal{M}$ and the degree of the polynomials may be greater than three. Sparsity is not necessary but admissible – if we can perturb Π linearly in $\mathbf{u}$, quadratically along the diagonal of $D_{\mathbf{uu}}\Pi$, and all the functions h_i, g_j in the constant terms.

The theorem below states that independent of the exact form or degree of the polynomials involved generically there will always appear only the same three different types of critical points.

We use the canonical topology for the finite dimensional space $\mathcal{M}$, i.e. a topology induced by the Euclidean distance in the coefficients of the polynomials involved.

Genericity Theorem *For $\Psi = (\Pi, g_1, \ldots, g_s, h_1, \ldots, h_q) \in \mathcal{M}$ let U_{ad} and Σ_{crit} be defined as in (28) and (31). Let $\mathcal{M}^*$ denote the set of those functions in $\mathcal{M}$ where each $(u, t) \in \Sigma_{crit}$ is of one of the Types 1,2,3 defined below. Then*

i) $\mathcal{M}^*$ contains an open and dense subset of $\mathcal{M}$;

ii) $\Sigma_{crit} \setminus \{$ points of Type 2 $\}$ is a finite union of one-dimensional C^∞-manifolds for all $\Psi \in \mathcal{M}^*$;

iii) the points of Type 2 and 3 form a finite subset of Σ_{crit} for all $\Psi \in \mathcal{M}^*$.

Remark: There are only three different Types of critical points in contrast to the five Types in [11]. This is due to the fact that the functions g_j, h_i are independent of t, so a small perturbation guarantees that the *LICQ* is satisfied for all $\mathbf{u}$ in the feasible set U_{ad}. □.

Type 1: nondegenerate critical point

We say that $(\overline{\mathbf{u}},\overline{t}) \in \Sigma_{crit}$ is of Type 1, if all the conditions $LICQ$ (30), $ND1$ (32) and $ND2$ (33) hold.

The variation of $(\overline{\mathbf{u}},\overline{\lambda},\overline{\mu})$ w.r.t. a small change in the parameter t can be followed by means of implicit function theorem for the function $F_{\mathcal{J}_0(\overline{\mathbf{u}})} = F$ with

$$F : \begin{pmatrix} \mathbf{u} \\ \lambda \\ \mu \\ t \end{pmatrix} \rightarrow \begin{pmatrix} D_u\Pi(\mathbf{u},\mathbf{t}) - \sum_{\mathbf{i}=1}^{\mathbf{q}} \lambda_\mathbf{i}\mathbf{D_u h_i}(\mathbf{u}) - \sum_{\mathbf{j}\in\mathcal{J}_0(\mathbf{u})} \mu_\mathbf{j}\mathbf{D_u g_j}(\mathbf{u}) \\ h_i(\mathbf{u}),\ \mathbf{i}=\mathbf{1},\ldots,\mathbf{q} \\ g_j(\mathbf{u}),\ \mathbf{j}\in\mathcal{J}_0(\mathbf{u}) \end{pmatrix} \tag{36}$$

with $F = (\overline{\mathbf{u}},\ \overline{\lambda},\ \overline{\mu},\ \overline{\mathbf{t}}) = \mathbf{0}$ and $F : I\!R^{n+q+s_0+1} \rightarrow I\!R^{n+q+s_0}$ where s_0 is the number of elements of $\mathcal{J}_0(\overline{\mathbf{u}})$, i.e. $s_0 = |\mathcal{J}_0(\overline{\mathbf{u}})|$.

The conditions $LICQ$ and $ND2$ guarantee the maximal rank of the Jacobian of F with respect to $(\mathbf{u},\lambda,\mu)$. The condition $ND1$ yields that the path $\mathbf{u}(t)$ for $|t-\overline{t}| \leq \epsilon$ does not leave the feasible set.

Classical smooth continuation techniques can be used in this case to follow the equilibrium configurations of the structure. No Lagrange multiplier and no eigenvalue of $D^2_{uu}L(\overline{\mathbf{u}},\overline{t})|_{T(\mathbf{u})}$ can change sign, so the topological type (minimum, saddle point, ...) and the number of active constraints are locally invariant (cf. [12]).

Type 2: Change in the active index set

The point $(\overline{\mathbf{u}},\overline{t}) \in \Sigma_{crit}$ is of Type 2, if $LICQ$ (30) and $ND2$ (33) are still satisfied – but for $\mathcal{J}_0(\overline{\mathbf{u}}) \neq \emptyset$ (w.l.o.g. $\mathcal{J}_0(\overline{\mathbf{u}}) = \{1,\ldots,p\}$) and exactly one of the Lagrange multipliers vanishes (w.l.o.g. $\mu_p = 0$), i.e. condition $ND1$ (32) is violated.

Furthermore, if the constraint function g_p were deleted, then $(\overline{\mathbf{u}},\overline{t})$ would be a point of Type 1 of the new problem. For this new problem the critical points can be locally represented by a differentiable function $\tilde{\mathbf{u}}(t)$ (cf. Type 1) for $|t-\overline{t}| \leq \epsilon$.

For $(\overline{\mathbf{u}},\overline{t})$ of Type 2 holds

$$\frac{d}{dt}(g_p \circ \tilde{\mathbf{u}})(\overline{t}) =: \gamma \neq 0 \tag{37}$$

This means, that the two curves that represent Σ_{crit} locally (one with g_p deleted, the other one with g_p treated as an equality constraint) intersect with nonzero angle. For more details, an explicit formula for γ and a geometric interpretation cf. [11] or [9], p.43.

Type 3: Quadratic turning point

The point $(\overline{\mathbf{u}},\overline{t}) \in \Sigma_{crit}$ is said to be of Type 3 if the conditions $LICQ$ (30) and $ND1$ (32) hold in $\overline{\mathbf{u}}$ and $(\overline{\mathbf{u}},\overline{\mathbf{t}})$ respectively, but *exactly one* eigenvalue of $D^2_uL(\overline{\mathbf{u}},\overline{\mathbf{t}})|_{\mathbf{T}(\overline{\mathbf{u}})}$ passes zero, so $ND2$ (33) is violated and the projection map $\phi(\mathbf{u},t) := t$ restricted to Σ_{crit} has a quadratic maximum or minimum in $(\overline{\mathbf{u}},\overline{\mathbf{t}})$. More details, characteristic numbers and a geometric interpretation can be found again in [11] and [9], p.44. The active index set of inequality constraints does not change at this point, thus smooth continuation techniques (analogous to the ones of section 4.1, cf. [9] or [11]) can be used to overcome this critical point.

4.4 Proof of the Genericity Theorem

We say that a given function $(\Pi, g_1, \ldots, g_s, h_1, \ldots, h_q) \in \mathcal{M}$ is an element of $\mathcal{M}^*(r)$, if and only if all those points $(\mathbf{u}, t) \in \Sigma_{crit}$ satisfying

$$\exists \lambda, \mu \quad : \quad F(\mathbf{u}, \lambda, \mu, t) = 0 \quad \text{and} \quad \|(\mathbf{u}, \lambda, \mu, t)\|_2^2 \leq r^2$$

are of Type 1, 2 or 3 (where F denotes the function from (36)).

We will show first, that the set $\mathcal{M}^*(r)$ is open and dense in $\mathcal{M}$.

4.4.1 Union of manifolds:

For each $\mathcal{J} \subseteq \{1, \ldots, s\}$ let $F_{\mathcal{J}}$ denote the function in (36) with $\mathcal{J}_0(u)$ replaced by $\mathcal{J}$. For each Function $F_{\mathcal{J}}$ there is by *Sard's theorem* an open and dense (semi–algebraic) set of regular values $d(\mathcal{J}) = (d(\mathcal{J})_i)_{1 \leq i \leq n+q+|\mathcal{J}|} \in I\!R^{n+q+|\mathcal{J}|}$. We can choose $d(\mathcal{J})$ arbitrarily close to zero and replace

$$\Pi(\mathbf{u}, t) \quad \text{by} \quad \Pi(\mathbf{u}, t) - \sum_{i=1}^{n} d(\mathcal{J})_i u_i, \tag{38}$$

and for all $1 \leq i \leq q$, $j \in \mathcal{J}$

$$h_i(\mathbf{u}) \quad \text{by} \quad h_i(\mathbf{u}) - d(\mathcal{J})_{n+1} \qquad \text{and} \qquad g_j(\mathbf{u}) \quad \text{by} \quad g_j(\mathbf{u}) - d(\mathcal{J})_{n+q+j}.$$

Then for $\|d(\mathcal{J})\|_2$ sufficiently small, the new problem still is an element of $\mathcal{M}$ by definition of $\mathcal{M}$.

But now the zero vector $\mathbf{0}$ is a *regular value* of $F_{\mathcal{J}}$ (related to the new functions Π, h_i, g_j), i.e. for all $(\mathbf{u}, \lambda, \mu, t)) \in \overline{B}(\mathbf{0}, r)$ holds

$$F_{\mathcal{J}}(\mathbf{u}, \lambda, \mu, t) = \mathbf{0} \quad \Rightarrow \quad \operatorname{rank}(DF_{\mathcal{J}}(\mathbf{u}, \lambda, \mu, t)) = n + q + |\mathcal{J}|. \tag{39}$$

Here $\mu = (\mu_j)$, $j \in \mathcal{J}$ denotes a real vector with $|\mathcal{J}|$ entries (depending on $\mathcal{J}$) in contrast to $\lambda = (\lambda_1, \ldots, \lambda_q)$. As a consequence of (39) the set

$$\{F_{\mathcal{J}} = 0\}_r := \{(\mathbf{u}, \lambda, \mu, t) \mid F_{\mathcal{J}}(\mathbf{u}, \lambda, \mu, t) = 0 \text{ and } \|(\mathbf{u}, \lambda, \mu, t)\|_2^2 \leq r^2\} \tag{40}$$

is the intersection of a *one dimensional* C^∞–manifold with the ball with radius r and center zero $\overline{B}(\mathbf{0}, r)$, if $q + |\mathcal{J}| \leq n$ – else the set $\{F_{\mathcal{J}_0(\mathbf{u})} = 0\}$ is *empty*, because there are too many constraints on $\mathbf{u}$ only.

The sets of regular values for the perturbation in (38) are open and dense, while there are only 2^s possibilities for the selection of the index set $\mathcal{J}$. Therefore we can find *one* arbitrary small perturbation of Π, h_i, g_j such that $\mathbf{0}$ is a regular value for *all* $F_{\mathcal{J}}$ *simultaneously.* In particular, the property (39) is *stable* under further small polynomial perturbations.

Studying only the last $q + |\mathcal{J}|$ components of $F_{\mathcal{J}}$, we see that a suitable, uniform choice of all $d(\mathcal{J})$ also guarantees the condition $LICQ$ for all $\mathbf{u} \in U_{ad}$. The property

(39) is stable i.e. it remains valid under further, sufficiently smooth polynomial perturbations in h_i, g_j as well. It also yields uniqueness of λ_i, μ_j.

Now we add to the function $F_{\mathcal{J}}$ components g_p, g_m with $p, m \in \{1, \ldots, s\} \backslash \mathcal{J}$. In analogy to (38) we obtain a small linear perturbation in Π and small constant perturbations in the h_i, g_j such that the zero vector is a regular value of the functions $\tilde{F}_{\mathcal{J},p}|_{\overline{\mathcal{B}}(0,r)}$ and $\widehat{F}_{\mathcal{J},p,m}|_{\overline{\mathcal{B}}(0,r)}$ with

$$\tilde{F}_{\mathcal{J},p}|_{\overline{\mathcal{B}}(0,r)}(\mathbf{u}, \lambda, \mu, t) := \begin{pmatrix} F_{\mathcal{J}}(\mathbf{u}, \lambda, \mu, t) \\ g_p(\mathbf{u}) \end{pmatrix}, \qquad p \notin \mathcal{J}$$

and

$$\widehat{F}_{\mathcal{J},p,m}|_{\overline{\mathcal{B}}(0,r)}(\mathbf{u}, \lambda, \mu, t) := \begin{pmatrix} F_{\mathcal{J}}(\mathbf{u}, \lambda, \mu, t) \\ g_p(\mathbf{u}) \\ g_m(\mathbf{u}) \end{pmatrix}, \qquad p, m \notin \mathcal{J} \text{ and } p \neq m$$

for all combinations of $\mathcal{J}, p, m$.

The function $\tilde{F}_{\mathcal{J},p}$ is a semi–algebraic function from $I\!R^{n+q+|\mathcal{J}|+1}$ into itself, so there are only finitely many solutions of the equation $\tilde{F}_{\mathcal{J},p}(\mathbf{u}, \lambda, \mu, t) = \mathbf{0}$, i.e. the set of points of *Type 2* is a *finite* subset of Σ_{crit}.

The function $\widehat{F}_{\mathcal{J},p}$ maps $I\!R^{n+p+|\mathcal{J}|+1}$ into $I\!R^{n+p+|\mathcal{J}|+2}$, so for $\mathbf{0}$ regular value the set of solutions of $\widehat{F}_{\mathcal{J},p}(\mathbf{u}, \lambda, \mu, t) = \mathbf{0}$, is empty. Consequently, it is impossible that two Lagrange multipliers in Σ_{crit} vanish simultaneously. Again, the two regularity conditions above are *stable* over $\overline{\mathrm{B}}(\mathbf{0}, r)$ under small polynomial perturbations.

In the sequel Π, h_i, g_j always denote the perturbed functions of (38), where $d(\mathcal{J})$ is chosen uniformly with respect to all the previous remarks.

4.4.2 Eigenvalues

For $(\overline{\mathbf{u}}, \overline{t}) \in \Sigma_{crit}$ with $F(\overline{\mathbf{u}}, \lambda, \mu, \overline{t}) = \mathbf{0}$, $F = F_{\mathcal{J}_0(\overline{\mathbf{u}})}$ (cf. (36)) and $(\mathbf{u}, \lambda, \mu, t) \in \overline{\mathcal{B}}(0, r)$ we know already that $LICQ$ holds in $\overline{\mathbf{u}}$ and that at most one of the Lagrange multipliers vanishes. There is left to study the rank conditions for the derivatives in more detail. Let $\widehat{L}$ denote the Lagrange function mapping $(\mathbf{u}, \lambda, \mu, t)$ to the first n components of F and assume w.l.o.g. that $\mathcal{J}_0(\mathbf{u}) = \{1, \ldots, p\}$, $p \leq n - q$. The derivative of F is a matrix of the form

$$DF(\mathbf{u}, \lambda, \mu, t) = \begin{pmatrix} D_u^2 \widehat{L}(\mathbf{u}, \lambda, \mu, t) & -B^T(u) & -\mathbf{p} \\ B(u) & 0 & 0 \end{pmatrix} \in I\!R^{(n+q+p)\times(n+q+p+1)}, \quad (41)$$

where the i-th row of $B(u)$ equals $Dh_i(u)$ if $i \leq q$ – else it equals $Dg_{i-q}(u)$. Due to $\operatorname{rank}[DF(\mathbf{u}, \lambda, \mu, t)] = n + q + p$ from (39) we have for

$$A(\mathbf{u}, \lambda, \mu, t) := D_{\mathbf{u},\lambda,\mu} DF(\mathbf{u}, \lambda, \mu, t) \quad \operatorname{rank}[A(\mathbf{u}, \lambda, \mu, t)] \geq n + p + q - 1. \quad (42)$$

Now we try to find a small parameter $\theta \in I\!R$ such that the perturbed problem with Π replaced by

$$\Pi_\theta(\mathbf{u}, t) := \Pi(\mathbf{u}, t) - \frac{\theta}{2} \sum_{i=1}^{n} u_i^2 \tag{43}$$

(with $F_\theta, A_\theta(\mathbf{u}, \lambda, \mu, t)$ defined accordingly) satisfies for all

$$(\mathbf{u}, \lambda, \mu, t) \in \{F_\theta = 0\}_r \quad \text{with} \quad \det(A_\theta(\mathbf{u}, \lambda, \mu, t)) = 0 \tag{44}$$

the condition

$$\forall \mathbf{v} \in I\!R^{n+p+q+1} \backslash \{\mathbf{0}\} \ : \ DF_\theta(\mathbf{u}, \lambda, \mu, t)\mathbf{v} = \mathbf{0} \Rightarrow (D\det(A_\theta))(\mathbf{u}, \lambda, \mu, t)\mathbf{v} \neq \mathbf{0}. \tag{45}$$

If $\det(A_\theta(\mathbf{u}, \lambda, \mu, t) = 0$ holds, then Condition (42) yields that *at most one* eigenvalue of A_θ passes zero. The Condition (45) guarantees that this happens with nonzero derivative along Σ_{crit}. Of course θ is chosen sufficiently small that all the regularity conditions discussed previously remain intact.

Let $W(\mathbf{u}) \in I\!R^{n \times (n-q-p)}$ be a matrix such that the columns of $W(\mathbf{u})$ form an orthonormal basis of $\mathcal{T}(\mathbf{u})$ from (35), the kernel of $B(\mathbf{u})$. The matrix $B(\mathbf{u})$ is of maximal rank due to $LICQ$. Using a new basis of $I\!R^{n+p+q}$ consisting of the columns of the matrix

$$\begin{pmatrix} W(\mathbf{u}) & 0 & B^T(\mathbf{u}) \\ 0 & \mathbf{E}_{q+p} & 0 \end{pmatrix}, \qquad \mathbf{E}_{q+p} \in I\!R^{(q+p)\times(q+p)} \text{ the unit matrix}$$

it is easily checked that

$$\det(A_\theta(\mathbf{u}, \lambda, \mu, t)) = -\det(B(\mathbf{u})B^T(\mathbf{u})) \cdot P(\mathbf{u}, \lambda, \mu, t, \theta) \tag{46}$$

with

$$P(\mathbf{u}, \lambda, \mu, t, \theta) = \det[W^T(\mathbf{u})(D_u^2 \widehat{L}(\mathbf{u}, \lambda, \mu, t) - \theta E_n)W(\mathbf{u})].$$

Seen as a function in θ the polynomial P is the characteristic polynomial of the matrix $Y(\mathbf{u}, \lambda, \mu, t) := [W^T(\mathbf{u})D_u^2 \widehat{L}(\mathbf{u}, \lambda, \mu, t)W(\mathbf{u})] \in I\!R^{(n-q-p)\times(n-q-p)}$. Due to $LICQ$ we have $\det(B(\mathbf{u})B^T(\mathbf{u})) \neq 0$, so the number of eigenvalues of $A_\theta(\mathbf{u}, \lambda, \mu, t)$ passing zero equals the number of eigenvalues of $Y(\mathbf{u}, \lambda, \mu, t)$ passing θ. Therefore the rank condition (42) yields for $(\mathbf{u}, \lambda, \mu, t)$ as in (44)

$$\frac{d}{d\theta} P(\mathbf{u}, \lambda, \mu, t, \mathbf{0}) \neq 0. \tag{47}$$

4.4.3 Critical points of Type 2:

For the new problem with Π replaced by Π_θ, $P(\mathbf{u}, \lambda, \mu, t, \theta)$ equals in fact the determinant of the matrix $D_u^2 L(\mathbf{u}, t)|_{T(u)}$ in condition $ND2$. There are only finitely many points where a Lagrange multiplier vanishes (cf. the end of Section *4.4.1*). For these points we can now obtain the nondegeneracy condition ND2 via an arbitrarily small perturbation in θ by (47). Again, the parameter θ can be chosen simultaneously for all sets $\mathcal{J} \subseteq \{1, \dots, s\})$ and we have stability over $\overline{\mathrm{B}}(\mathbf{0}, r)$. Let $(\overline{\mathbf{u}}, \overline{t})$ be one of these points. There is left to check condition (37) (cf. Type 2, w.l.o.g. $\mu_p = 0$). Let $\tilde{\mathbf{u}}(t)$ denote the differentiable curve parametrizing the set of critical points of the problem where the constraint g_p has been deleted. Then we can choose an arbitrary small regular value d_p of the function $g_p \circ \tilde{\mathbf{u}}$ and replace $g_p(\mathbf{u})$ by $g_p(\mathbf{u}) - d_p$ (constant perturbation is admissible in). Then the *transversality condition* (37) is satisfied (for an explicit formula for γ cf. [11] or [9]). This completes the discussion of Type 2. The candidates for the perturbations above can be picked from open sets (thus stability).

4.4.4 Critical ponts of Type 3:
For abbreviation we put (for $\mathbf{d} \in \mathbb{R}^{n+q+p}$, $\mathcal{J}$ fixed)

$$\mathcal{F}(\mathbf{d}, \theta, \mathbf{u}, \lambda, \mu, t) := (F_\theta(\mathbf{u}, \lambda, \mu, t) - \mathbf{d}, P(\mathbf{u}, \lambda, \mu, t, \theta)).$$

Then the condition (45) is equivalent to the condition

$$\mathcal{F}(\mathbf{d}, \theta, \mathbf{u}, \lambda, \mu, t) = \mathbf{0} \Rightarrow \operatorname{rank}(D_{u,\lambda,\mu,t}\mathcal{F}(\mathbf{d}, \theta, \mathbf{u}, \lambda, \mu, t) = n + q + p + 1 \qquad (48)$$

for all $(\mathbf{u}, \lambda, \mu, t) \in \overline{\mathcal{B}}(0, r)$.

The computation of suitable parameters $\mathbf{d}, \theta$ near zero such that (48) holds is based on the idea of Thom's transversality theorem. For $(\mathbf{u}_0, \lambda_0, \mu_0, t_0) \in \overline{\mathcal{B}}(0, r)$ satisfying

$$\mathcal{F}(\mathbf{0}, 0, \mathbf{u}_0, \lambda_0, \mu_0, t_0) = \mathbf{0}$$

differentiation of $\mathcal{F}$ with respect to $\mathbf{d}$, θ yields

$$D_{d,\theta}\mathcal{F}(\mathbf{0}, 0, \mathbf{u}_0, t_0) = \begin{pmatrix} -\mathbf{E}_{n+q+p} & \begin{matrix} -\mathbf{u} \\ \mathbf{0} \end{matrix} \\ \mathbf{0}^T & g \end{pmatrix} \in \mathbb{R}^{(n+q+p+1)\times(n+q+p+1)} \qquad (49)$$

with $g = \frac{d}{d\theta}P(\mathbf{u}_0, \lambda_0, \mu_0, t_0, 0) \neq 0$ by (47).

So the implicit function theorem can be applied locally. We cover the compact set

$$\Omega := \{(\mathbf{u}, \lambda, \mu, t, \mathbf{0}_{n+q+p+1})|\ (\mathbf{u}, \lambda, \mu, t) \in \overline{\mathcal{B}}(0, r),\ \mathcal{F}(\mathbf{0}\}$$

by finitely many small open sets $\mathcal{O}_i$ where the assumptions of the implicit function theorem hold. Let $\pi(\mathcal{O}_i)$ denotes the projection of $\mathcal{O}_i$ to the first $n + q + p + 1$ components. The sets $\mathcal{O}_i$ are chosen such that for all $(\mathbf{u}, \lambda, \mu, t) \in \pi(\mathcal{O}_i)$ semi-algebraic C^∞-functions $\mathbf{d}_i, \theta_i$ are defined with

$$\mathbf{d}_i(\mathbf{u}(i), \dots, t(i), \mathbf{0}) = \mathbf{0}, \quad \theta_i(\mathbf{u}(i), \dots, t(i), \mathbf{0}) = 0 \qquad (50)$$

and

$$\mathcal{F}(\mathbf{d}_i(\mathbf{u}, \dots, t), \theta_i(\mathbf{u}, \dots, t), \mathbf{u}, \lambda, \mu, t) = 0, \qquad \forall(\mathbf{u}, \dots, t) \in \pi(\mathcal{O}_i). \qquad (51)$$

For all i the set of regular values of $\Psi_i := (\mathbf{d}_i, \theta_i)$ is open and dense - we can find one regular value $(\mathbf{d}_{reg}, \theta_{reg})$ for all Ψ_i arbitrarily close to zero. Then for

$$(\overline{\mathbf{u}}, \overline{\lambda}, \overline{\mu}, \overline{t}) \in \pi(\mathcal{O}_i) \quad \text{with} \quad F(\mathbf{d}_{reg}, \theta_{reg}, \overline{\mathbf{u}}, \overline{\lambda}, \overline{\mu}, \overline{t}) = \mathbf{0} \tag{52}$$

holds

$$\Psi_i(\overline{\mathbf{u}}, \overline{\lambda}, \overline{\mu}, \overline{t}) = (\mathbf{d}_{reg}, \theta_{reg}) \quad \text{and} \quad \text{rank}[D\Psi_i(\overline{\mathbf{u}}, \ldots, \overline{t})] = n + q + p + 1. \tag{53}$$

Differentiating (51) with respect to $\mathbf{d}$, θ then yields for this $(\overline{\mathbf{u}}, \ldots, \overline{t})$

$$D_{\mathbf{d},\theta}\mathcal{F}(\mathbf{d}_{reg}, \theta_{reg}, \overline{\mathbf{u}}, \ldots, \overline{t}) D\Psi(\overline{\mathbf{u}}, \ldots, \overline{t}) + D_{\mathbf{u},\lambda,\mu,t}\mathcal{F}(\mathbf{d}_{reg}, \ldots, \overline{t}) = \mathbf{0} \tag{54}$$

The first two matrices have maximal rank due to (49), the definition of $\mathcal{O}_i$ and regularity of $(\mathbf{d}_{reg}, \theta_{reg})$. So the second matrix is of maximal rank $(n + q + p + 1)$ also and this gives the assertion (48) inside of the union of all the sets $\pi(\mathcal{O}_i)$.

Now $\mathcal{N} := \overline{\mathcal{B}}(0, r) \setminus \cup_i \pi(\mathcal{O}_i)$ is compact and

$$\forall (\mathbf{u}, \ldots, t) \in \mathcal{N} \; : \; \mathcal{F}(\mathbf{0}, 0, \mathbf{u}, \ldots, t) \neq 0.$$

Continuity of $\mathcal{F}$ then yields for $(\mathbf{d}_{reg}, \theta_{reg})$ sufficiently small

$$\forall (\mathbf{u}, \ldots, t) \in \mathcal{N} \; : \; \mathcal{F}(\mathbf{d}_{reg}, \theta_{reg}, \mathbf{u}, \ldots, t), \neq 0 \tag{55}$$

which proves the assertions (45) and (48).

These imply for $(\overline{\mathbf{u}}, \ldots, \overline{t}) \in \overline{\mathcal{B}}$ with $\mathcal{F}(\mathbf{d}_{reg}, \theta_{reg}, \overline{\mathbf{u}}, \ldots, \overline{t}) = \mathbf{0}$ that the projection map $\phi_t(\mathbf{u}, t) := t$ restricted to Σ_{crit} near $(\overline{\mathbf{u}}, \overline{t})$ has a quadratic maximum or minimum in $(\overline{\mathbf{u}}, \overline{t})$ (cf. [11], p. or [9], p.45 for a characteristic number sign(β) indicating maximum or minimum). Below we only discuss the unconstrained setting $(q = p = 0)$. The general case can be reduced to this case via local coordinate transformation (see [13]).

Let $(\overline{\mathbf{u}}, \overline{t})$ be a critical point as in (48) (with $q = p = 0$) and

$$f(\mathbf{u}, t) := \Pi(\mathbf{u}, t) - \mathbf{d}_{deg}^T \mathbf{u} - \frac{\theta_{deg}}{2} \sum_{i=1}^{n} u_i^2.$$

Due to the rank condition (42) we can parametrize the solution of $D_{\mathbf{u}} f(\mathbf{u}, t) = 0$ locally via implicit function theorem (w.l.o.g. as a function in u_1) by

$$\mathbf{w}(u_1) = (u_1, u_2(u_1), \ldots, u_n(u_1), t(u_1)) \quad \text{with} \quad \mathbf{w}(\overline{u}_1) = (\overline{\mathbf{u}}, \overline{t}).$$

Then the derivative $\dot{\mathbf{w}}(u_1)$ is the tangent vector to Σ_{crit} in $\mathbf{w}(u_1)$, and for all u_1 near $\overline{u}_1$

$$[DD_{\mathbf{u}} f(\mathbf{w}(u_1))] \dot{\mathbf{w}}(u_1) = \mathbf{0} \tag{56}$$

For $(\overline{\mathrm{u}},\overline{t}) \in \Sigma_{deg}$ the rank condition (39) is still valid so

$$\mathrm{rank}[D_u^2 f(\overline{\mathrm{u}},\overline{t})] = n-1, \quad \mathrm{rank}[DD_u f(\overline{\mathrm{u}},\overline{t})] = n \quad \Rightarrow \quad t'(\overline{u}_1) = 0 \tag{57}$$

where $t'(u_1)$ is the last component of $\dot{\mathrm{w}}(u_1)$. Now (57) implies that $(\overline{\mathrm{u}},\overline{t})$ is a critical point of ϕ_t.

Let $\omega(u_1)$ denote the first n components of $\dot{\mathrm{w}}(u_1)$. Then (57) and (56) furthermore yield that $\omega(\overline{u}_1)$ is an eigenvector for the zero eigenvalue of $D_u^2 f(\overline{\mathrm{u}},\overline{t})$. Via an additional, linear independent equation we can obtain locally a smooth curve $\mathrm{v}(u_1)$ of eigenvalues of $D_{\mathbf{u}}^2 f(\mathrm{w}(u_1))$ with

$$\mathrm{v}(\overline{u}_1) = \omega(\overline{u}_1) \quad \text{and} \quad \mathrm{v}^T(u_1) D_{\mathbf{u}}^2 f(\mathrm{w}(u_1))\mathrm{v}(u_1) = \xi(u_1)\|\mathrm{v}(u_1)\|_2^2,$$

where $\xi(u_1)$ denotes the eigenvalue of $D_{\mathbf{u}}^2 f(\mathrm{w}(u_1))$ passing zero in $\overline{u}_1$. Then there is a differentiable function h such that

$$\omega(u_1)^T [D_{\mathbf{u}}^2 f(\mathrm{w}(u_1))]\omega(u_1) = \xi(u_1)\|\omega(u_1)\|_2^2 + (u_1 - \overline{u}_1)^2 h(u_1). \tag{58}$$

Differentiation of (56) multiplied by $\dot{\mathrm{w}}^T(u_1)$ from the left hand side, together with the relation (58) and $\xi'(\overline{u}_1) \neq 0$ due to (45) yield $t''(\overline{u}_1) \neq 0$ – i.e. we have a *quadratic* turning point. This finishes the proof of the auxiliary condition that $\mathcal{M}^*(r)$ is open and dense in $\mathcal{M}$.

4.4.5 Extension of the results to the whole space $I\!R^n$

The set $\mathcal{M}$ is a complete metric space, so it is a Baire space. Thus the intersection

$$\mathcal{M}^* = \bigcap_{r=1}^{\infty} \mathcal{M}^*(r) \tag{59}$$

is still a *dense* subset of $\mathcal{M}$ (cf. [15], p. 387). Due to this semi-algebraic nature of the problem there exsists a uniform upper bound on the number of connected components of the sets of points 1, 2, 3 independent of r (cf. [1]). The set $\mathcal{M}^*$ also is a semi-algebraic set in the coefficients of the polynomials - so if $\mathcal{M}^*$ is dense one can find (by means of a stratification, cf. [1]) a subset of $\mathcal{M}^*$ that is open and dense in $\mathcal{M}$. *q.e.d.*

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